Hydroinformatik - SoSe 2026

UW-BHW-414-12: Grundlagen der Hydromechanik

Prof. Dr.-Ing. habil. Olaf Kolditz

[1]Helmholtz Centre for Environmental Research – UFZ, Leipzig

[2]Technische Universität Dresden – TUD, Dresden

[3]Center for Advanced Water Research – CAWR

[4]TUBAF-UFZ Center for Environmental Geosciences – C-EGS, Freiberg / Leipzig

Dresden, 12.06.2026

Zeitplan: Hydroinformatik I+II

Sommersemester 2026: Stand: 06.04.2026

Nr.	KW	Datum	ID	Thema
01+02	16	17.04.2026	UW-BHW-414-01/02	Einführung in die Vorlesung, Umweltinformatik
03	16	17.04.2026	UW-BHW-414-03	Werkzeuge, Hello World (in C++)
05	17	24.04.2026	UW-BHW-414-04	Selbststudium: Software-Installationen
07	19	08.05.2026	UW-BHW-414-05	Objekt-Orientierte Programmierung: C++, Klassen
09	20	15.05.2026	UW-BHW-414-06	Programmiersprache Python
11	21	22.05.2026	UW-BHW-414-07/08	Modellierung, Digitalisierung - Wasser 4.0
00	22	29.05.2026		Vorlesungsfreie Woche
13	23	05.06.2026	UW-BHW-414-09/10	KI, Maschinelles Lernen, Neuronale Netzwerke
15	24	12.06.2026	UW-BHW-414-11/12	Kontinuumsmechanik, Hydromechanik
17	25	19.06.2026	UW-BHW-414-I	Differentialgleichungen, Näherungsverfahren
19	26	26.06.2026	UW-BHW-414-J	Finite-Differenzen, explizite Verfahren
21	27	03.07.2026	UW-BHW-414-K	Finite-Differenzen, implizite Verfahren
23	28	10.07.2026	UW-BHW-414-L	Gerinnehydraulik, Grundwasserhydraulik
25	29	17.07.2026	UW-BHW-414-M	Grundwasserhydraulik
27	30	24.07.2026	UW-BHW-414-N	Zusammenfassung, Klausurvorbereitung

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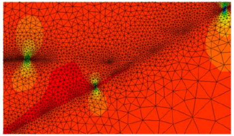
Fahrplan für heute ...

1. UW-BHW-414-12: Grundlagen der Hydromechanik
   - Semesterplan

Konzept

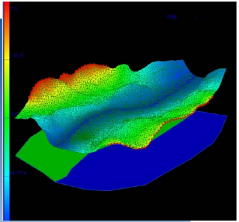
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

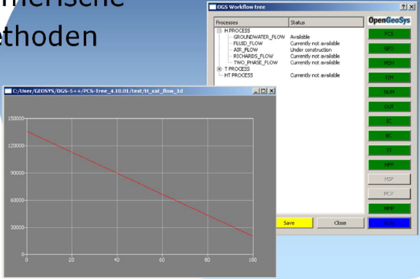
Mechanik

Anwendung



Numerische

Methoden



Programmierung

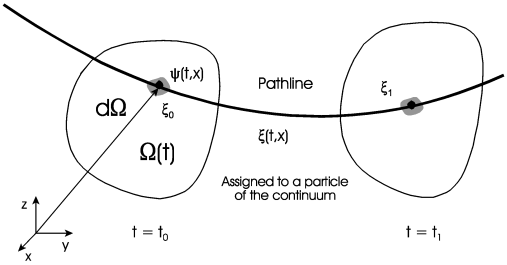
Visual C++

Prozessverständnis

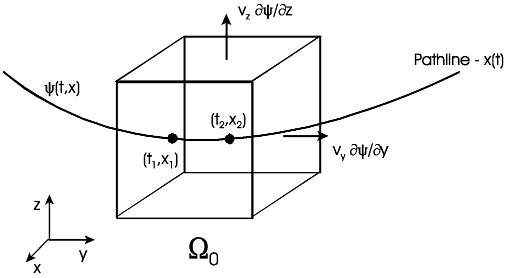
Fahrplan für heute

- Erhaltungsgrößen
- Massenerhaltung
- Fluidmassenerhaltung
- Diffusion
- Impulserhaltung
- Spannungen
- Fluiddruck
- Strömungsprobleme

Lagrange



Euler



$$\frac{d}{dt}\int_\Omega \psi \, d\Omega = \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \oint_{\partial\Omega} \mathbf{\Phi}^\psi \cdot d\mathbf{S} \quad (1)$$

$$= \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \int_\Omega \nabla \cdot \mathbf{\Phi}^\psi d\Omega$$

$\lim d\Omega \rightarrow 0$

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot \mathbf{\Phi}^\psi \quad (2)$$

$$= \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi)$$

$$= Q^\psi$$

HA 02#2020: Komponentenschreibweise $\nabla \cdot (\mathbf{v}\psi)$, $\nabla \cdot (\mathbf{D}^\psi \nabla\psi)$

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{3}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	$\mathbf{m}$	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \tag{4}$$

The differential equation of mass conservation in divergence form becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \tag{5}$$

Partial differentiation of the above equation gives

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0 \tag{6}$$

(Phase) Mass Conservation

Using the material (or convective) derivative the mass conservation equation can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v} \tag{7}$$

Note, above convective form of mass conservation equation becomes zero only for incompressible flows, i.e.

$$\frac{\partial \rho}{\partial t} = 0 \tag{8}$$

requires divergence-free flow.

$$\nabla \cdot \mathbf{v} = 0 \tag{9}$$

From eqn. (6) results that the above expression is the continuity equation for a homogeneous fluid ($\rho = \text{const}$).

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Divergenzfreie Strömung

$$\nabla \cdot \mathbf{v} = 0 \tag{10}$$

Links:

- https://de.wikipedia.org/wiki/Stromfunktion
- https://www.ingenieurkurse.de/stroemungslehre/ebene-stroemungen/quelle-und-senke-divergenz.html

```python
import numpy as np
import matplotlib.pyplot as plt
# set up a normalized grid:
dim= 20
xarray= np.arange(-dim,dim)
yarray= np.arange(-dim,dim)
# (fluid) flow from a source at L to a sink at -L:
L = dim/2
x,y = np.meshgrid(xarray,yarray)
vx = (x-L)/((x-L)**2+y**2) - (x+L)/((x+L)**2 +y**2)
vy = y/((x-L)**2+y**2) - y/((x+L)**2 +y**2)
# plot the flow lines:
plt.figure()
plt.quiver(x,y, vx, vy, pivot='mid')
plt.xlabel("$x$-axis")
plt.ylabel("$y$-axis")
plt.axis('equal')
plt.show()
```

Listing: Python code for divergence-free flow - div v

Source: University of Auckland

Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{11}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	$\mathbf{m}$	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

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Momentum Conservation

$\psi = \rho\mathbf{v}$

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\rho\mathbf{v}d\Omega + \oint\limits_{\partial\Omega}\mathbf{\Phi}^{m}\cdot d\mathbf{S} = \int\limits_{\Omega}\rho\mathbf{f}d\Omega \tag{12}$$

Flux term: The advective momentum flux is defined as

$$\mathbf{\Phi}^{m} = (\rho\mathbf{v})\otimes\mathbf{v} = (\rho\mathbf{v})\mathbf{v} \tag{13}$$

$$\mathbf{F} = \int\limits_{\Omega}\rho\mathbf{f}d\Omega = \int\limits_{\Omega}\rho(\mathbf{f}^{e}+\mathbf{f}^{i})d\Omega = \underbrace{\int\limits_{\Omega}\rho\mathbf{f}^{e}d\Omega}_{\text{External forces}} + \underbrace{\oint\limits_{\partial\Omega}\boldsymbol{\sigma}:d\mathbf{S}}_{\text{Internal forces}} \tag{14}$$

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Substituting now flux and source terms of momentum we obtain

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\rho\mathbf{v}d\Omega+\oint\limits_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S})=\int\limits_{\Omega}\rho\mathbf{f}^{e}d\Omega+\oint\limits_{\partial\Omega}\boldsymbol{\sigma}d\mathbf{S} \tag{15}$$

Applying the Gauss-Ostrogradskian theorem to the surface integrals

$$\begin{aligned}\oint\limits_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S}) &= \int\limits_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})\,d\Omega\\ \oint\limits_{\partial\Omega}\boldsymbol{\sigma}:d\mathbf{S} &= \int\limits_{\Omega}\nabla\cdot\boldsymbol{\sigma}\,d\Omega\end{aligned} \tag{16}$$

Momentum Conservation

The differential form of the momentum conservation law is then

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \rho\mathbf{f}^e + \nabla\cdot\boldsymbol{\sigma} \tag{17}$$

The above equation is now extended by partial integration

$$\begin{aligned} \rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + (\rho\mathbf{v})\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot(\rho\mathbf{v}) \\ = \rho[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}] + \mathbf{v}[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})] \\ = \rho\mathbf{f}^e + \nabla\cdot\boldsymbol{\sigma} \end{aligned} \tag{18}$$

Using the mass conservation equation (5) and dividing by ρ we obtain

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla\cdot\boldsymbol{\sigma} \tag{19}$$

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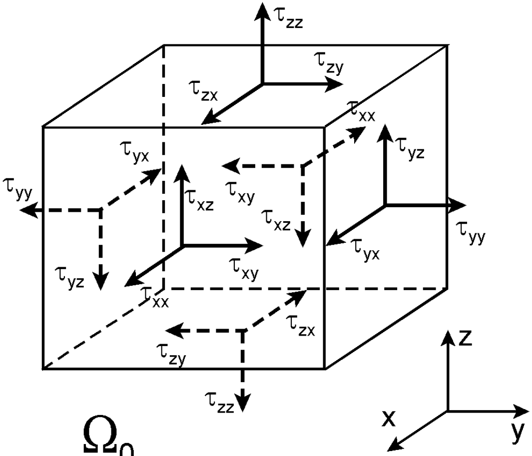
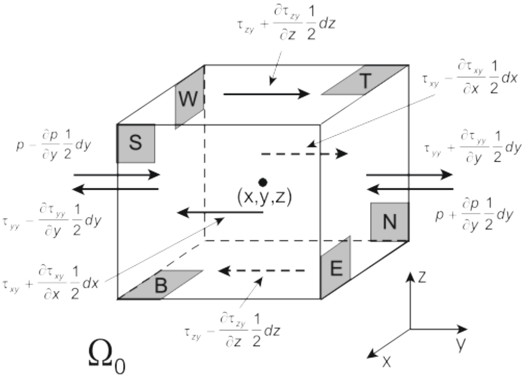
Momentum Conservation: Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \quad , \quad \boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{20}$$



Momentum Conservation: Stress Tensor

$$\boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{21}$$



$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{22}$$

In index notation the above vector equation is written as

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}) \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= g + \frac{1}{\rho}(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})
\end{aligned} \tag{23}$$

with $u = v_x, v = v_y, w = v_z$ and $\mathbf{f}^e = \mathbf{g}$.

Flow Equations - Systematic

Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \tag{24}$$

Navier-Stokes Equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}} \tag{25}$$

Euler Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p \tag{26}$$

Stokes Equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{27}$$

Darcy Equations

$$0 = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{28}$$

(Component) Mass Conservation

$$\psi = \rho_k = C_k \tag{29}$$

$$\frac{dC_k}{dt} = \frac{\partial C_k}{\partial t} + \nabla \cdot (\mathbf{v} C_k) - \nabla \cdot (\mathbf{D_k} \nabla C_k) = Q_k \tag{30}$$

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Übungen und Hausaufaben

- EX: Übungen
- HA: Hausaufgaben

Übungen
Mechanik

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Übungen: Hydroinformatik

GitHub-Repo: `https://github.com/OlafKolditz/Hydroinformatik-2026`

- EX20: Jupyter Notebook
- EX21: Kontinnumsmechanik: Skalarprodukt
- EX22: Hydromechanik: Divergenzfreie Strömung
- EX23: Analytische Lösung: Elliptische Gleichung
- EX24: Analytische Lösung: Parabolische Gleichung (Diffusion)
- EX25: Analytische Lösung: Transportgleichung (ADE)
- EX26: Finite-Differenzen-Methode (FDM) explizit
- EX27: Finite-Differenzen-Methode (FDM) implizit
- EX28: Gerinnehydraulik

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Hausaufgaben

1. Skalarprodukt: Schreiben sie das Skalarprodukt $\nabla \cdot \mathbf{v}$ in Komponentenschreibweise.
2. Mechanik: Was ist $\mathbf{v} \cdot \nabla\psi$? Physikalische Bedeutung des Terms
3. Mechanik: Was ist Φ^ψ? Physikalische Bedeutung des Terms
4. Hydromechanik: Komponentenschreibweise $\nabla \cdot (\mathbf{v}\psi)$
5. Hydromechanik: Komponentenschreibweise $\nabla \cdot (\mathbf{D}^\psi \nabla\psi)$
6. Analytik: Prüfen Sie die Gültigkeit einer der Lösungen für die Diffusionsgleichung: (20), (21), (22) (siehe Vorlesung 5 $>>$ HyBHW-S2-01-V05).
7. Analytik: Stellen Sie die ausgewählte analytische Lösung für die 1-D parabolische Differentialgleichung unter Verwend der Übung EX06-parabolische-gleichung-1D.py dar. Ergänzen Sie Ihren Namen oder Matrikelnummer mit dem Befehl `plt.title("Name oder Matrikelnummer")`.
8. Numerik: Darstellung der numerischen Lösung (explizite FDM) für die 1-D parabolische Differentialgleichung (EX08-fdm-explicit-python). Produzieren Sie eine stabile und instabile Lösung.
9. Numerik: Darstellung der numerischen Lösung (implizite FDM) für die 1-D parabolische Differentialgleichung (EX09-fdm-implicit-python). Produzieren Sie die stationäre Lösung.

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Hausaufgaben: Hydroinformatik

Beispiel: Aufgaben 7-9

- zum Internet-Repository gehen (Webseite)
- Python-File editieren (Matrikel-Nummer oder Name)
- Programme zum Rechnen und Darstellen ausführen
- Ergebnis (Abbildung) in die Hausaufgaben einfügen

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